

Avalanche mixing of granular solids in a rotating 2D drum and discrete mapping

S. N. Dorogovtsev*

A.F.Ioffe Physico-Technical Institute, 194021 St.Petersburg, Russia

Evolution of mixing of granular solids in a slowly rotated 2D drum is considered as a discrete mapping. The rotation is around the axis of the upright drum which is filled partially, and the mixing occurs only at a free surface of a material. The most simple cases that demonstrate clearly the essence of such a type of mixing are studied analytically. We calculate the characteristic time of the mixing and the distribution of the mixed material over the drum.

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By the term "avalanche mixing"¹ one calls the mixing of granular solids that proceeds *only* at a free surface layer. Mixing occurs in a slowly rotating upright drum (we shall consider only a two dimensional drum) which is filled partially (see fig. 1). Granules mix in avalanches falling from the upper half of a free surface to the lower half. In spite of the simplicity of the process, nature of the avalanche mixing is nontrivial and experimental mixing patterns look striking¹. Nevertheless, nature of the avalanche mixing can be understood using purely geometrical approach¹⁻⁵. Note that though mixing and segregation in rotated drums is studied intensively⁶⁻²⁰, the avalanche mixing is of special interest, because of clearness of the problem formulation.

If one assumes that the difference between the angles of repose and marginal stability is little (the case of the finite difference is studied thoroughly by B. A. Peratt and J. A. Yorke^{3,5}) it is possible to describe the mixing analytically^{2,4}. (Note that small granules of the different fractions are distinguished only by color.) The fastest possible avalanche mixing takes place if fractions mix completely in avalanches, so the granular material is always mixed at the lower half of a free surface. The experiment¹ appears to be surprisingly

close to this extremal regime⁴. Below we shall consider an opposite extremal situation in which granules, while flowing along a free surface are not mixed at all. One may ask: is there any mixing in such a situation at all? We shall give a positive answer. The mixing exists (apart from a center part of the drum, of course), and it is rather efficient. Indeed, the fractions will be finally mixed! We shall describe dynamics of such a mixing.

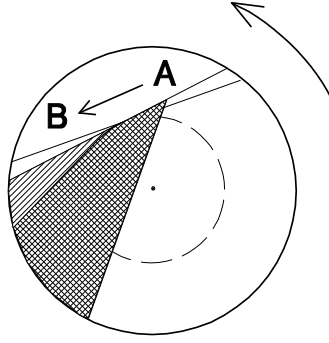


FIG. 1. The avalanche mixing scheme. For an infinitely small turn of the drum, the granules of different fractions flow from sector A to sector B , undergoing mixing. The free surface of the material is tilted at the angle of repose all the time. The pure fractions are shown by the black and white colours. Regions of mixed material are denoted by the grey colour. The degree of mixing is not shown. The material inside of the dashed circle will be never mixed. The result of the avalanche mixing depends on the way in which the granules flow from sector A to sector B .

Then there are two possible cases. In the first one, granules with the linear coordinate x at the upper half of a free surface (we assume that at the free surface midpoint $x = 0$) fall to the point $x' = x$ at the lower half, so in fact the surface wedge is not changed after avalanches (see Fig. 2,b).

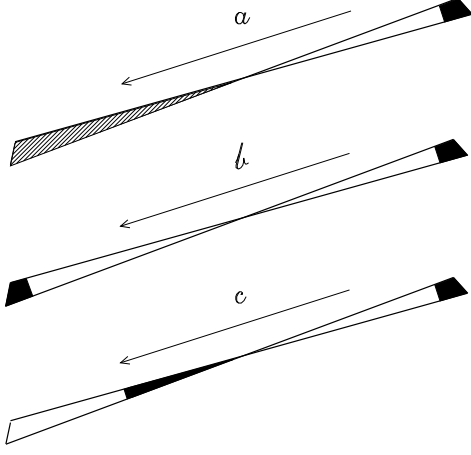


FIG. 2. Limit regimes of the evolution of granules while flowing from the upper half of a free surface to the lower half.

a – the material mixes completely in the avalanches;

b – after the flow, the material appears to be inverted, and upper granules become lower ones – the $x' = x$ transformation of the wedges (x is a linear coordinate at a free surface, counted from the midpoint of the free surface in both directions).

c – the material slips as a whole from the upper half of a free surface to the lower half, so lower granules stay to be lower ones after the slip. Using the condition of volume conservation one obtains easily the $x' = \sqrt{\sin^2 \theta - x^2}$ transformation of the wedges. Here the angle θ characterizes the relative volume of an empty space in the drum.

In the second case, the material from the upper wedge slips to the lower one as a whole (Fig. 2,c). The drum radius is unit, and the measures of the amount of the material are the angles $\theta \leq \pi/2$ for more then half filled drum and $\vartheta = \pi - \theta \leq \pi/2$ for less then half filling (see Fig. 1). Thus, the coordinates of the lower and the upper points of a free surface will be $\sin \theta$ or $\sin \vartheta$. Then, accounting for volume conservation, one describes the transformation of the coordinates as $x' = \sqrt{\sin^2 \theta - x^2}$ (or $x' = \sqrt{\sin^2 \vartheta - x^2}$) in these situations.

Introducing the scaled coordinate $z \equiv x/\sin \theta$ or $z \equiv x/\sin \vartheta$ we get for the cases under consideration: $z' = z$ and $z' = \sqrt{1 - z^2}$.

Let us describe by the number k events of appearing of some granule at the lower half of a free surface. The corresponding coordinate of the granule at the lower part of a free surface and time will be (x_k, t_k) . (The angle of the drum turn plays the role of the time.) Transformation of this pair describes completely the mixing, since all the granules move as a whole in the bulk of the material. We shall study this discrete mapping.

We start with consideration of the $z' = z$ transformation case for less then half filled drum. One can see from Fig. 1 that the coordinate is unchanged during the mapping and $t_{k+1} = t_k + 2 \arctan(x_k / \cos \vartheta)$.

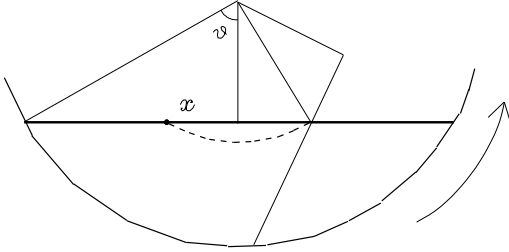


FIG. 3. The avalanche mixing in the event of a less then half filled drum and the $x' = x$ transformation of the wedges. A free surface is shown to be horizontal, since our results do not depend on the angle of repose. The angle $\vartheta \leq \pi/2$ characterizes the relative volume of the granular material. If a granule is at the point x at the left half of a free surface, it will always stay at a dashed circle sector.

Thus, one obtains the following trivial discrete mapping²¹:

$$(z_{k+1}, t_{k+1}) = (z_k, t_k + 2 \arctan(z_k \tan \vartheta)) . \quad (1)$$

One may study the simplest case of an infinitely thin layer of the black fraction over the white fraction at the initial moment, that is $(z_0, t_0) = (z, 0)$. Then the black material will

evolve in the following manner:

$$(z_k, t_k) = (z, 2k \arctan(z \tan \vartheta)) . \quad (2)$$

Thus, the black granules will be at the point z of a free surface at moments $t_k(z)$, $k = 0, 1, 2 \dots$ (see Fig. 4).

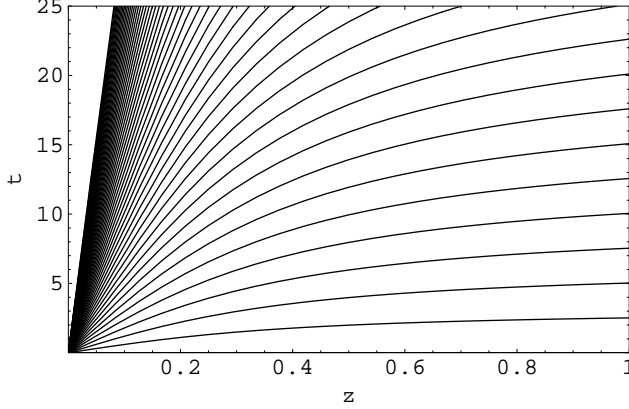


FIG. 4. Evolution of a thin layer of the black fraction for a less than half filled drum in the case of the $z' = z$ transformation ($z \equiv /x \sin \vartheta$). $\vartheta = 0.4\pi$. (See Eq. (2)). Crossings of the line $t = const$ with the curves correspond to the points at the lower half of a free surface in which black granules appears at the time t .

Formally speaking, there will be no true mixing of the material at any time, but near the fixed z the distance between black granule layers are diminished with time. When, in the vicinity of z , this distance becomes to be of the order of the granule diameter, the material is in fact mixed in this area. A very simple geometrical consideration gives for this distance:

$$d = \frac{2}{t} \frac{z \sin \vartheta (1 + z^2 \tan \vartheta) \arctan^2(z \tan \vartheta)}{\{[1 + (1 + z^2 \tan^2 \vartheta) \arctan(z \tan \vartheta)/t]^2 + z^2 \tan^2 \vartheta\}^{1/2}} . \quad (3)$$

If now d is the relative diameter of granules (recall that the drum radius equals 1) then one obtains the corresponding characteristic time

$$t_{mix} = \arctan(z \tan \vartheta) \left\{ \left[\frac{4z^2 \sin^2 \vartheta}{d^2} (1 + z^2 \tan \vartheta) \arctan^2(z \tan \vartheta) - z^2 \tan^2 \vartheta \right]^{1/2} - 1 \right\} . \quad (4)$$

The expression is most simple in the following limit cases. If $z \tan \vartheta \ll 1$,

$$t_{mix} = \frac{2z^3}{d} \tan^2 \vartheta \sin \vartheta . \quad (5)$$

If $z \tan \vartheta \gg 1$,

$$t_{mix} = \frac{\pi^2 z^2}{2d} \tan \vartheta \sin \vartheta . \quad (6)$$

Finally, the black fraction from the point z will be distributed homogeneously over the dotted line shown in Fig. 3. If initially the black fraction occupies the sector of the $\gamma \rightarrow 0$ angle at the lower half of a free surface, then the concentration of the black fraction in the mixture will be

$$c(z) = \frac{\gamma}{2 \arctan(z \tan \vartheta)} , \quad (7)$$

so the mixing differs greatly for different z . As it can be seen from Eq. (6), when $\vartheta \rightarrow \pi/2$, $t_{mix} \rightarrow \infty$, and any mixing is practically absent (see Fig. 4).

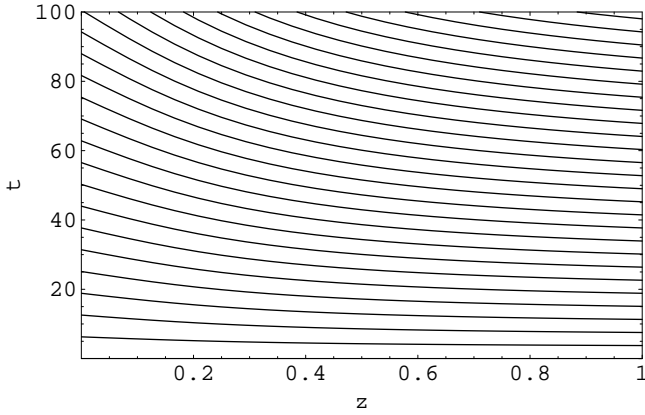


FIG. 5. Same as Fig. 4 for a more than half filled drum in the case of the $z' = z$ transformation ($z \equiv \sin \theta$). $\theta = 0.4\pi$. (See Eq. (8)).

The case of a more than half filled drum $\theta \equiv \pi/2 - \vartheta \leq \pi/2$ can be considered in the same manner. In the event $z' = z$ one writes

$$t_k = 2k[\pi - \arctan(z \tan \theta)] \quad (8)$$

(see Fig. 5) – cf Eq. (2). The material in the point z will be mixed to a "homogeneous" state with the black fraction concentration

$$c(z) = \frac{\gamma}{2\pi - 2 \arctan(z \tan \theta)} , \quad (9)$$

in a time

$$t_{mix} = [\pi - \arctan(z \tan \theta)] \left\{ \left[\frac{4z^2 \sin^2 \theta}{d^2} (1 + z^2 \tan^2 \theta) [\pi - \arctan(z \tan \theta)]^2 - z^2 \tan^2 \theta \right]^{1/2} - 1 \right\}. \quad (10)$$

Thus, if $z \tan \theta \gg 1$,

$$t_{mix} = \frac{\pi^2}{2d} \tan \theta \sin \theta \quad (11)$$

and in the event of $z \tan \theta \ll 1$

$$t_{mix} = \frac{2\pi^2 z}{d} \sin \theta. \quad (12)$$

Note, that Eqs (10) and (12) are valid only if there are many crossings of the black fraction trace with the lower half of a free surface, i.e. if $t_{mix} \gg 2\pi(2\pi/2\theta)$. Thus, as it follows from Eq. (12), the θ angle should be $\theta \gg \sqrt{d/z}$. Otherwise $t_{mix} \sim 2\pi^2/\theta$.

We see that there is a minimum of t_{mix} at some intermediate filling value. A full dependence of t_{mix} on ϑ (or rather $t_{mix}^{-1}(\vartheta)$) is shown in Fig. 6. It somewhat resembles the characteristic mixing time behavior in the case of a full mixing in the wedges¹, though the meaning of t_{mix} is quite different now.

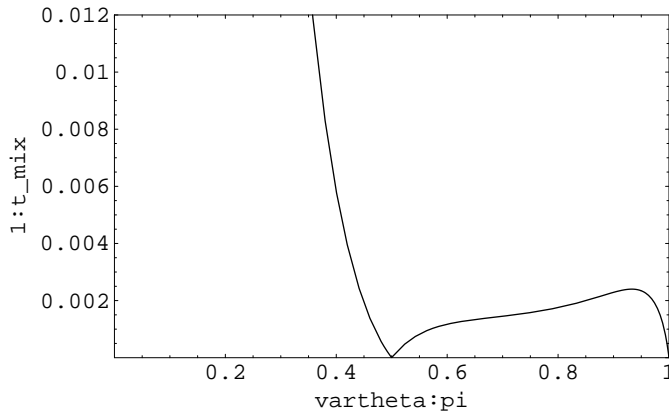


FIG. 6. t_{mix}^{-1} vs ϑ/π . The later one characterizes the relative volume of the granular material: the drum is empty, half filled, or full when $\vartheta = 0, \pi/2$, or π , correspondingly. The crossover region near $\vartheta \sim 1$ is shown by guess-work. $z = 0.5, d = 0.01$.

Now we proceed with the case of a less than half filled drum and the $z' = \sqrt{1 - z^2}$ transformation. One see that

$$(z_{k+1}, t_{k+1}) = \left(\sqrt{1 - z_k^2}, t_k + 2 \arctan(z_k \tan \vartheta) \right), \quad (13)$$

so if initially $(z_0, t_0) = (z, 0)$, then

$$\begin{aligned} (z_{2n}, t_{2n}) &= \left(z, 2n \arctan(z_k \tan \vartheta) + 2n \arctan(\sqrt{1 - z_k^2} \tan \vartheta) \right) \\ (z_{2n+1}, t_{2n+1}) &= \left(\sqrt{1 - z^2}, 2(n+1) \arctan(z_k \tan \vartheta) + 2n \arctan(\sqrt{1 - z_k^2} \tan \vartheta) \right), \end{aligned} \quad (14)$$

where $n = 0, 1, 2, \dots$. Therefore,

$$\begin{aligned} t_{2n} &= 2n \arctan(z \tan \vartheta) + 2n \arctan(\sqrt{1 - z^2} \tan \vartheta) \\ t_{2n+1} &= 2n \arctan(z \tan \vartheta) + (2n + 1) \arctan(\sqrt{1 - z^2} \tan \vartheta) \end{aligned} \quad (15)$$

(see Fig. 7).

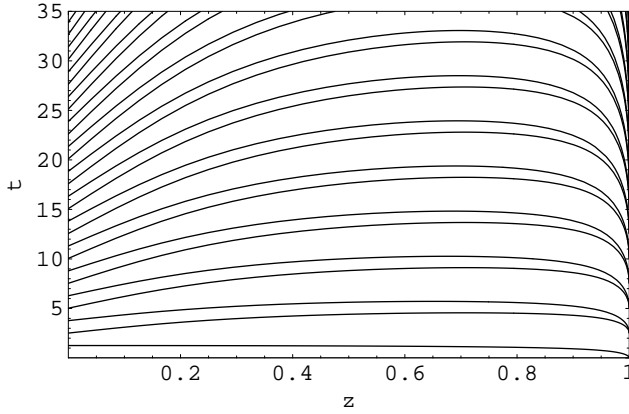


FIG. 7. Same as Figs 4 and 5 for a less than half filled drum and the $z' = \sqrt{1 - z^2}$ transformation of the wedges. $\theta = 0.4\pi$. (See Eq. (15)).

Now there is a point $z = 1/\sqrt{2}$ in which the fractions will be never mixed, and mixing in fact occurs in the vicinity of $z = 0$ or $z = 1$. The region of poor mixing is broaden as the drum filling tends to the one half level. The averaged concentration of the black fraction is

$$c(z) = \frac{\gamma}{\arctan(z \tan \vartheta) + \arctan(\sqrt{1 - z^2} \tan \vartheta)} \quad (16)$$

(cf Eq. (7)).

In the same manner can be considered case of the $z' = \sqrt{1 - z^2}$ transformation for a more than half filled drum. One can obtain easily

$$\begin{aligned} t_{2n} &= 4n\pi - 2n \arctan(z \tan \theta) - 2n \arctan(\sqrt{1 - z^2} \tan \theta) \\ t_{2n+1} &= (2n + 1)2\pi - 2n \arctan(z \tan \theta) - (2n + 1) \arctan(\sqrt{1 - z^2} \tan \theta) \end{aligned} \quad (17)$$

(see Fig. 8).

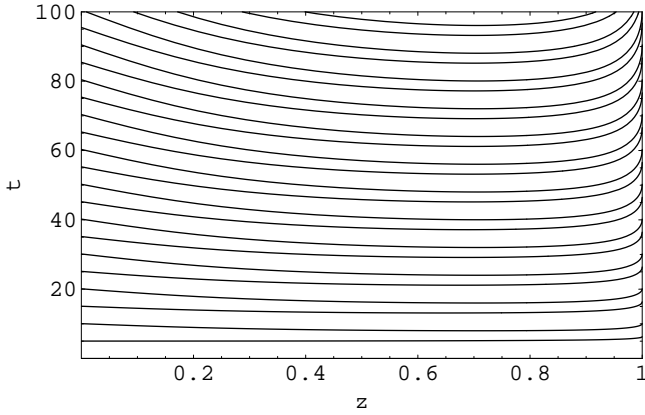


FIG. 8. Same as Figs 4, 5, and 7 for a more than half filled drum and the $z' = \sqrt{1 - z^2}$ transformation of the wedges. $\theta = 0.4\pi$. (See Eq. (17)).

The averaged concentration of the black fraction will be

$$c(z) = \frac{\gamma}{2\pi - \arctan(z \tan \theta) - \arctan(\sqrt{1 - z^2} \tan \theta)} \quad (18)$$

(cf Eq. (9)). Note that in the event of the $z' = \sqrt{1 - z^2}$ transformation, mixing is far poorer than for the $z' = z$ transformation.

In conclusion, using discrete mapping approach, we considered analytically two limit regimes of avalanche mixing, when the mixing in the wedges is absolutely absent. Nevertheless, finally, the material appears to be rather effectively mixed. The experimental situation¹ – the mixing of salt grains – is between full mixing in wedges and what we call the $z' = z$ transformation of the wedges (see the experimental patterns of the mixing in the paper of G. Metcalfe *et al*). There was a question^{2,4}: why is the experiment so close to the extremal (the fastest!) regime of avalanche mixing, in which granules fully mix in avalanches? Why do the theory for this extremal regime give such a surprising coincidence with the mixing

time values observing in the experiment? Now we can see the reason. As we have shown in the present communication, the dependence of the quantity characterizing the time of mixing on the drum filling (see Fig. 6) in the case of the $z' = z$ transformation appears to be rather similar to the behavior of mixing time in the fastest possible for avalanche mixing regime. Thus the experiment, which is between this two limit cases with a similar behavior, can be described well.

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* E-mail: sn@dor.ioffe.rssi.ru

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